Tropical Geometry and Commutative Algebra for Semirings

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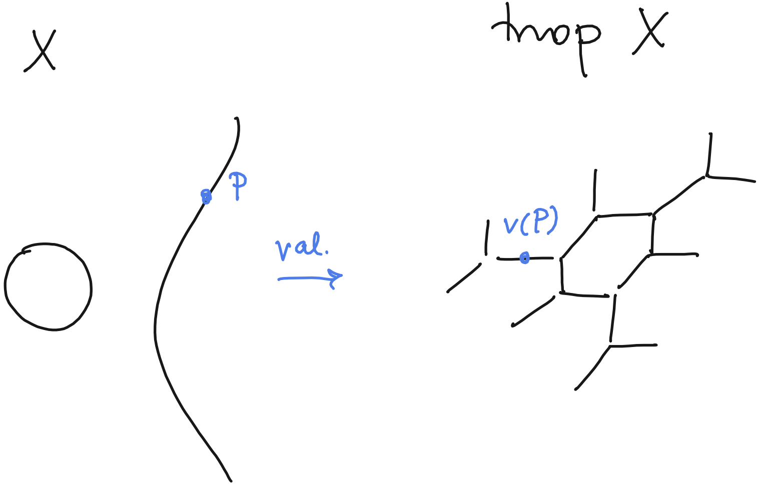

Goal: Understand X by understanding *trop*(X).

Algebraic Geometry

$v: k \to \mathbb{R} \cup \{\infty\} = (\mathbb{T}, \underset{\min}{+}, \underset{+_{\mathbb{R}}}{\times})$

$v(ab) = v(a) + v(b)$

$v(a+b) \geq \min\{v(a), v(b)\}$

$v(0) = \infty$

Commutative Algebra:

ideal $I \subseteq k[x_1, \dots x_n]$

$I \ni f = \sum c_u \boldsymbol{x}^{\boldsymbol{u}}$

$\longmapsto$

Tropical Algebra:

$\operatorname{trop}(f) = \min\{v(c_u) + x.u\}$

$\operatorname{trop}(f) \in \mathbb{T}[x_1 \cdots x_n]$

$\operatorname{trop}(I) = \{\operatorname{trop}(f) \mid \forall f \in I\}$

Geometry:

$V(I) = \{P \in k^n : f(P) = 0, \forall f \in I\}$

Tropical Geometry:

$\operatorname{trop}(V(I)) =$

$\{P \in \mathbb{T}^n \mid f(P)$ is either ∞ or min is attained twice$\}$

Motivation

1. Applications to algebraic geometry:
    - Moduli spaces of curves
    - Enumerative geometry (Gromov-Witten invariants)
    - Brill-Noether theory
    - Mirror Symmetry
2. Outside algebraic geometry:
    - Math biology (phylogentic trees)
    - Economics
    - Neural Networks
    - etc.

Pros/Cons of these methods

Pros:

- New set of combinatorial tools.

Cons:

- Lose algebraic information (this is a degeneration)
- Hard to work in higher dimensions (beyond curves)

Sub-Goal: Salvage enough algebra and create an intrinsic theory.

Remarks

$$" \{ \mathbb{R} \cup \{\infty\}, \min, +_{\mathbb{R}} \}$$

- $\mathbb{T}$ is additively idempotent

$$\forall a \in \mathbb{T}, \quad a + a = a.$$

- There is no "-" (semifield)

- $\mathbb{T}[x_1, \ldots, x_n]$ is not cancellative, not UFD, ...

We will try to understand:

- Quotients (coordinate rings of affine varieties)

  $k[x_1 \dots x_n]/I$

- Prime ideals(?)

- Dimension

In the ring case:

- Let I be an ideal in a ring R, we define $C_I = \langle (a,0), \forall a \in I \rangle$, then

$$R/I := R/C_I.$$

ker of cong.

- One-to-one correspondence between I and C_I

In the semiring case:

not the case

$\mathbb{T}[x,y]/x\sim y \cong \mathbb{T}[x]$.

ker of $\langle x \sim y \rangle$ is trivial

nothing other than $0_{\mathbb{T}}$ is $\sim$ to $0_{\mathbb{T}}$.

Ring multiplication (pair version)

$a \sim b \iff (a,b) \in C_I \iff (a-b,0) \in C_I$

Let $a - b, c - d \in I$, i.e. (a, b) and $(c, d) \in C_I$ $\iff (a-b) \in I.$

Note that $\boxed{(a, b) \cdot (c, d)} \in C_I$

$\iff (a - b, 0) \cdot (c - d, 0) \in C_I$

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$\iff (ac+bd-(ad+bc),0) \in C_I$
$\iff (ac+bd, ad+bc) \in C_I$

Definition

Let C be a congruence on R, and let $\alpha=(a,b)$ and $\beta=(c,d)$. The **twisted product** $\alpha*\beta=(a,b)*(c,d)=(ac+bd, ad+bc)$.

Primes

Let R be a ring or a semiring.

P is a **prime ideal** of R if whenever $ab \in P$ then $a \in P$ or $b \in P$.

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Theorem (Joó-M'17)

If R is an additively idempotent semiring, then P is a prime congruence if and only if R/P is cancellative and P is irreducible.

$ab = cb$

$\Rightarrow$ either $b = 0$

or $a = c$.

$P = A \cap B$

$\Rightarrow P = A$ or $P = B$.

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If R is an additively idempotent semiring, then P is a prime congruence if and only if R/P is cancellative and totally ordered.

Primes on $\mathbb{B}[x_1, \ldots, x_n]$ and $\mathbb{T}[x_1, \ldots, x_n]$

- The primes on the polynomial semiring or Laurent polynomial semiring correspond to matrices.
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$\mathbb{B} = \{0, 1\}$

$1 + 1 = 0.$

Example

Let $U = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, *that defines the prime* $P(U)$ *in* $\mathbb{B}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$.

We would like to compare the following monomials in $\mathbb{B}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]/P(U)$. *Let* $m_1 = x^2y^3z$ *and* $m_2 = x^3yz^2$.

$$\mathbf{n}_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \ \mathbf{n}_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \text{ and } U\mathbf{n}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \ U\mathbf{n}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$$U\mathbf{n}_1 - U\mathbf{n}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \text{ hence } m_1 > m_2.$$

$\mathbb{T}[x_1 \dots x_n]$.

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Definition JM'17

The (Krull) **dimension** of R is the number of strict inclusions of prime congruences in a chain of maximal length.

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Example

dim $\mathbb{B} = 0$; *dim* $\mathbb{T} = 1$; *dim* $\mathbb{B}[x] = 1$; *dim* $\mathbb{T}[x_1, \ldots, x_n] = n + 1$

$\{0, 1\}$

We can do these via explicit computation.

$\dim \mathbb{T} = 1$

$a \sim 1_{\mathbb{T}}$ when $a \neq 0_{\mathbb{T}}$

Dimension Theory

Theorem (Joó-M'18)

Let R be an additively idempotent semiring, then

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The proof goes by passing to the semifield of fractions of R, which has the same dimension as R!

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Remark

If R is a Noetherian ring, then $\dim R[x] = \dim R + 1$. Otherwise, $\dim R + 1 \leq \dim R[x] \leq 2\dim R + 1$.

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Theorem (F. Alarcón and D. Anderson'94)

If we define dimension in terms of ideals then $\dim \mathbb{B}[x] = \infty$.

Let $I \in \mathbb{T}[x_1, \ldots, x_n]$ be an **ideal** then

min

$$V(I) = \{a \in \mathbb{T}^n : f(a) \text{ attains its maximum at least twice}, \forall f \in I\}.$$

or $f(a) = \infty$

Let $C \in \mathbb{T}[x_1, \ldots, x_n]^2$ be a **congruence** then

$$V(C) = \{a \in \mathbb{T}^n : f(a) = g(a), \forall (f, g) \in C\}.$$

$f \sim g$

Towards Geometry: Tropical vanishing locus (1)

Let $I \in \mathbb{T}[x_1, \ldots, x_n]$ be an **ideal** then

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Question: What can we say about $V(P)$ when P is a prime ideal or congruence of $\mathbb{T}[x_1, \ldots, x_n]$?

Towards Geometry: Tropical vanishing locus (2)

Theorem (Joó-M'21)

Let C be a prime **congruence** *or a prime* **ideal** *on $\mathbb{T}[x_1, \ldots, x_n]$. Then:*

$$V(P) = \emptyset \text{ or } V(P) = \text{point}.$$

if P is a prime ideal

& P is "tropical"

then $P = \mathbb{T}[x_1 \ldots x_n]$.

Convergent power series

semiring.

Let M be a toric monoid. Let $\mathcal{O}_P$ denote the set of "convergent power series" at the point P, i.e. P is a prime congruence and $\mathcal{O}_P$ is a subset of $\{f = \sum_{u \in M} c_u \chi^u, c_u \in S\}$.

Theorem (Friedenberg-M'21)

If $P \in \mathbb{T}[M]$ with trivial kernel, then

$$\dim \mathcal{O}_P = \dim \mathbb{T}[M].$$

We an inequality for subsemifields of $\mathbb{T}$ and the formula involves the rank of the residue semi. field.

Towards Geometry (2)

For an affine algebraic variety:

$$\dim\ V(I) = \dim\ k[x_1, \ldots, x_n]/I.$$

The Structure Theorem for Tropical Geometry:

$$\dim\ V(I) = \dim\ \underbrace{\mathrm{trop}\ V(I)}_{\text{polyh. complex}}.$$

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$$\dim\ \text{trop}\ V(I) \underset{?}{=} \dim\ \mathbb{T}[x_1, \ldots, x_n]/???.$$

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Yes!

Bend Congruences

Let I be an ideal of $k[x_1, \ldots x_n]$, then
$\underbrace{trop(I)}_{J} = \{trop(f) : f \in I\}$ $\subseteq \mathbb{T}[x_1, \ldots x_n]$.

Let J be an ideal of $\mathbb{T}[x_1, \ldots x_n]$
$bend(\cancel{f}\, g) = \{g \sim g_{\hat{i}} : \ i \in supp(g)\}$
$bend(J) = \{bend(g) : g \in J\}$
$\uparrow$ bend congruences, due to Giansiracusa^2'16

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Example

$g = x + y + z$, *then* $bend(g) = \{g \sim x+y \sim x+z \sim y+z\}.$

Towards Geometry (3)

The $\mathbb{T}$-points of $\underbrace{Spec\mathbb{T}[x_1, \ldots, x_n]/bend(trop(I))}$ are $trop(V(I))$.

The $\mathbb{T}$-points of $Spec\mathbb{T}[x_1, \ldots, x_n]/bend(trop(I))$ are $trop(V(I))$.

Theorem

Let I be an ideal of $k[x_1, \ldots x_n]$, then

$$dim\ trop\ V(I) = dim\left(\mathbb{T}[x_1, \ldots, x_n]/bend(trop(I))\right) - 1.$$

Recall that $dim\ \mathbb{T} = 1$.

polyh. complex.

Remark

Ideals of the type trop$(I) \subseteq \mathbb{T}[x_1, \ldots, x_n]$ are referred to as "tropicalized ideals". They are a proper subset of the set of **tropical ideals**, *introduced by Maclagan-Rincón'17.*

- *A tropical ideal $J \subseteq \mathbb{T}[x_1, \ldots, x_n]$ is almost never finitely generated (neither is trop(I) for $I \in k[x_1, \ldots x_n]$).*
- *The congruence bend(J) is also almost never finitely generated.*
- *However, $trop(V(I)) = V(trop(I)) = \cap_{f \in T} V(trop(f))$, where T is finite.*
- *$HF(I) = HF(trop(I))$, where HF is the Hilbert function.*

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- *However, $trop(V(I)) = V(trop(I)) = \cap_{f \in T} V(trop(f))$, where T is finite.*
- *$HF(I) = HF(trop(I))$, where HF is the Hilbert function.*

The degree of the Hilbert Polynomial agrees with the dimension of the polyhedral complex and the Krull dimension we introduced!

-1

Thank you for your attention!